3.2. Mean Value Theorem

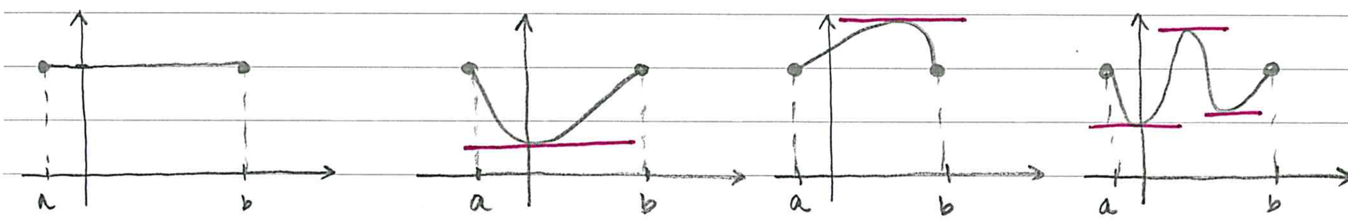
Rolle's Thm.: Suppose a function f satisfies:

(1). f is continuous on $[a, b]$;

(2). f is differentiable on (a, b);

(3). $f(a) = f(b)$.

Then there is a number $c \in (a, b)$ such that $\boxed{f'(c) = 0.}$



Mean Value Thm.: Suppose a function f satisfies:

(1). f is continuous on $[a, b]$;

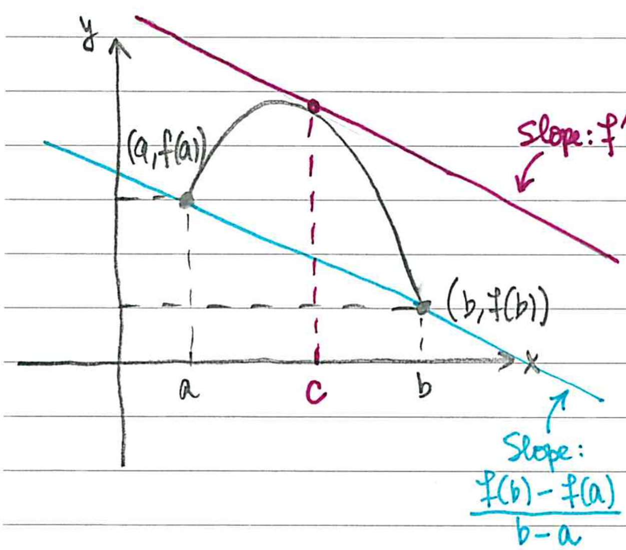
(2). f is differentiable on (a, b);

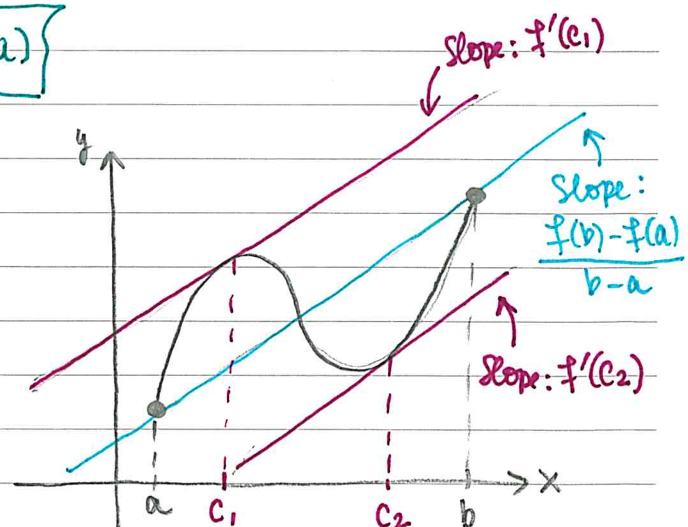
Then there is a number $c \in (a, b)$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

or, equivalently:

$$\boxed{f(b) - f(a) = f'(c)(b - a)}$$





PROOF: Apply Rolle's Theorem to a new function

$$h(x) := f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

* h is continuous on $[a,b]$ & diff'ble on (a,b);
* $h(a) = 0$; $h(b) = 0$

$\Rightarrow$ there is $c \in (a,b)$ s.t. $h'(c) = 0$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \Rightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

∎

VIP Corollary #1: If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b).

Pf.: Let any $x_1 < x_2$ in (a,b), and apply MVT to f on $[x_1, x_2]$:

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{0}(x_2 - x_1) \text{ for some } c \in (x_1, x_2)$$

$\Rightarrow f(x_1) = f(x_2)$ for all $x_1 < x_2$ in (a,b)

$\Rightarrow f$ is constant on (a,b). ∎

VIP Corollary #2: If $f'(x) = g'(x)$ for all $x \in (a,b)$, then $(f-g)$ is constant on (a,b), so $f(x) = g(x) + C$, where C is a constant.

Pf.: Apply Corollary #1 to the function $h = f - g$. ∎

Example 1: $f(x) = 5-2x^2$ on $[-6,7]$.

* f is continuous on $[-6,7]$ and differentiable on $(-6,7)$.
* by MVT, there is $c \in (-6,7)$ s.t.

$$f'(c) = \frac{f(7)-f(-6)}{7-(-6)} = \frac{-2\cdot 13}{13} = -2$$

* What is c?

$$f'(x) = -4x$$

$$f'(c) = -2 \Rightarrow -4c = -2 \Rightarrow c = \frac{1}{2}$$

Example 2: $f(x) = x + \frac{10}{x}$ on $[\frac{1}{3}, 3]$.

* f is continuous on $[1/3, 3]$ and diff'ble on $(1/3, 3)$.
* by MVT, there is $c \in (1/3, 3)$ s.t.

$$f'(c) = \frac{f(3)-f(1/3)}{3-1/3} = \frac{3+\frac{10}{3}-\frac{1}{3}-30}{\frac{8}{3}} = \frac{-24}{8/3} = -9$$

* What is c?

$$f'(x) = 1 - \frac{10}{x^2}$$

$$f'(c) = -9 \Rightarrow 1 - \frac{10}{c^2} = -9 \Rightarrow \frac{c^2-10}{c^2} = -9 \Rightarrow c^2 - 10 = -9c^2$$

$$\Rightarrow 10c^2 - 10 = 0 \Rightarrow c = \pm 1 \Rightarrow c = 1$$

($c = -1$ is not in $(1/3, 3)$).

Example 3: $f(x) = x^2 - 8x + 7$ on $[0,8]$

* f is continuous on $[0,8]$ & diff'ble on $(0,8)$;
* $f(0) = f(8) = 7$, so we can apply Rolle's Thm. : there is $c \in (0,8)$ s.t. $f'(c) = 0$.
* What is c?

$$f'(x) = 2x - 8$$

$$f'(c) = 0 \Rightarrow 2c - 8 = 0 \Rightarrow c = 4 \in (0,8)$$

Example 4: $f(x) = \sqrt{6-x}$ on $[-6, 6]$.

* f is continuous on $[-6,6]$ and diff'ble on $(-6,6)$.
* by MVT, there is $c \in (-6,6)$ s.t.

$$f'(c) = \frac{f(6) - f(-6)}{6-(-6)} = \frac{0 - \sqrt{12}}{12} = \frac{-1}{\sqrt{12}} = \frac{-1}{2\sqrt{3}}$$

* What is c?

$$f'(x) = \frac{-1}{2\sqrt{6-x}}$$

$$f'(c) = \frac{-1}{\sqrt{12}} \Rightarrow \frac{-1}{2\sqrt{6-c}} = \frac{-1}{2\sqrt{3}} \Rightarrow \sqrt{6-c} = \sqrt{3} \Rightarrow 6-c = 3$$

$$\Rightarrow c = 3$$

Example 5: $f(x) = 5\sqrt{x} - 3x$ on $[9, 16]$.

* by MVT, there is $c \in (9,16)$ s.t.:

$$f'(c) = \frac{f(16) - f(9)}{16-9} = \frac{5 \cdot 4 - 48 - (5 \cdot 3 - 27)}{7} = \frac{-16}{7}$$

* find c? $f'(x) = \frac{5}{2\sqrt{x}} - 3$

$$f'(c) = -\frac{16}{7} \Rightarrow \frac{5}{2\sqrt{c}} - 3 = -\frac{16}{7} \Rightarrow \frac{5}{2\sqrt{c}} = \frac{5}{7} \Rightarrow 2\sqrt{c} = 7 \Rightarrow \sqrt{c} = \frac{7}{2}$$

$$c = \frac{49}{4}$$